

# Reconstructing sparse exponential polynomials from samples: Stirling numbers and Hermite interpolation

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**Abstract.** Prony’s method, in its various concrete algorithmic realizations, is concerned with the reconstruction of a sparse exponential sum from integer samples. In several variables, the reconstruction is based on finding the variety for a zero dimensional radical ideal. If one replaces the coefficients in the representation by polynomials, i.e., tries to recover sparse exponential polynomials, the zeros associated to the ideal have multiplicities attached to them. The precise relationship between the coefficients in the exponential polynomial and the multiplicity spaces are pointed out in this paper.

## 1 Introduction

In this paper we consider an extension of what is known as *Prony’s problem*, namely the reconstruction of a function

$$f(x) = \sum_{\omega \in \Omega} f_{\omega}(x) e^{\omega^T x}, \quad 0 \neq f_{\omega} \in \Pi, \quad \omega \in \Omega \subset (\mathbb{R} + i\mathbb{T})^s, \quad (1)$$

from multiinteger samples, i.e., from samples  $f(A)$  of  $f$  on a subgrid  $A$  of  $\mathbb{Z}^s$ . Here, the function  $f$  in (1) is assumed to be a sparse *exponential polynomial* and the original version of Prony’s problem, stated in one variable in [14], is the case where all  $f_{\omega}$  are *constants*. Here “sparsity” refers to the fact that the cardinality of  $\Omega$  is small and that the frequencies are either too unstructured or too irregularly spread to be analyzed, for example, by means of Fourier transforms.

Exponential polynomials appear quite frequently in various fields of mathematics, for example they are known to be exactly the homogeneous solutions of partial differential equations [8,9] or partial difference equations [15] with constant coefficients.

I learned about the generalized problem (1) from a very interesting talk of Bernard Mourrain at the 2016 MAIA conference, September 2016. Mourrain [13] studies extended and generalized Prony problems, especially of the form (1), but also for log-polynomials, by means of sophisticated algebraic techniques

like Gorenstein rings and truncated Hankel operators. This paper approaches the problem in a different way, following the concepts proposed in [16,17], namely by factorizing a Hankel matrix in terms of Vandermonde matrices; this factorization, stated in Theorem 4 has the advantage to give a handy criterion for sampling sets and was a useful tool for understanding Prony's problem in several variables, cf. [17]. Moreover, the approach uses connections to the description of finite dimensional kernels of multivariate convolutions or, equivalently, the homogeneous solutions of systems of partial difference operators.

This relationship that connects finite differences, the Taylor expansion and the Newton form of interpolation on the multiinteger grid, can be conveniently expressed in terms of multivariate Sterling numbers of the second kind and will be established in Section 2. In Section 3, this background will be applied to define the crucial "Prony ideal" by means of a generalized Hermite interpolation problem that yields an *ideal projector*. The aforementioned factorization, stated and proved in Section 4 then allows us to directly extend the algorithms from [16,17] which generate ideal bases and multiplication tables to the generalized problem without further work. How the eigenvalues of the multiplication tables relate to the common zeros of the ideal in the presence of multiplicities is finally pointed out and discussed in Section 5.

The notation used in this paper is as follows. By  $\Pi = \mathbb{C}[x_1, \dots, x_s]$  we denote the ring of polynomials with complex coefficients. For  $A \subset \mathbb{N}_0^s$  we denote by  $\Pi_A = \text{span}\{(\cdot)^\alpha : \alpha \in A\} \subset \Pi$  the vector space spanned by the monomials with exponents in  $A$ . The set of all multiindices  $\alpha \in \mathbb{N}_0^s$  of *length*  $|\alpha| = \alpha_1 + \dots + \alpha_s$  is written as  $\Gamma_n := \{\alpha \in \mathbb{N}_0^s : |\alpha| \leq n\}$  and defines  $\Pi_n = \Pi_{\Gamma_n}$ , the vector space of polynomials of total degree at most  $n$ .

## 2 Stirling numbers and invariant spaces of polynomials

The classical *Sterling numbers* of the second kind, written as  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$  in Karamata's notation, cf. [7, p. 257ff], can be defined as

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} := \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n. \quad (2)$$

One important property is that they are *differences of zero* [6], which means that

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \frac{1}{k!} \Delta^k 0^n := \frac{1}{k!} (\Delta^k (\cdot)^n) (0).$$

Since this will turn out to be a very useful property, we define the multivariate Sterling numbers of the second kind for  $\nu, \kappa \in \mathbb{Z}^s$  as

$$\left\{ \begin{smallmatrix} \nu \\ \kappa \end{smallmatrix} \right\} := \frac{1}{\kappa!} (\Delta^\kappa (\cdot)^\nu) (0) = \frac{1}{\kappa!} \sum_{\gamma \leq \kappa} (-1)^{|\kappa| - |\gamma|} \binom{\kappa}{\gamma} \gamma^\nu, \quad (3)$$

with the convention that  $\left\{ \begin{smallmatrix} \nu \\ \kappa \end{smallmatrix} \right\} = 0$  if  $\kappa \not\leq \nu$  where  $\alpha \leq \beta$  if  $\alpha_j \leq \beta_j$ ,  $j = 1, \dots, s$ . Moreover, we use the usual definition

$$\binom{\kappa}{\gamma} := \prod_{j=1}^s \binom{\kappa_j}{\gamma_j}.$$

The identity in (3) follows from the definition of the difference operator

$$\Delta^\kappa := (\tau - I)^\kappa, \quad \tau p := [p(\cdot + \epsilon_j) : j = 1, \dots, s], \quad p \in \Pi,$$

by straightforward computations. From [15] we recall the degree preserving operator

$$Lp := \sum_{|\gamma| \leq \deg p} \frac{1}{\gamma!} \Delta^\gamma p(0) (\cdot)^\gamma, \quad p \in \Pi, \quad (4)$$

which has a representation in terms of Sterling numbers: if  $p = \sum_\alpha p_\alpha (\cdot)^\alpha$ , then

$$Lp := \sum_{|\gamma| \leq \deg p} (\cdot)^\gamma \sum_{|\alpha| \leq \deg p} p_\alpha \frac{1}{\gamma!} (\Delta^\gamma (\cdot)^\alpha)(0) = \sum_{|\gamma| \leq \deg p} \left( \sum_{|\alpha| \leq \deg p} \left\{ \begin{smallmatrix} \alpha \\ \gamma \end{smallmatrix} \right\} p_\alpha \right) (\cdot)^\gamma,$$

that is,

$$(Lp)_\alpha = \sum_{\beta \in \mathbb{N}_0^s} \left\{ \begin{smallmatrix} \beta \\ \alpha \end{smallmatrix} \right\} p_\beta, \quad \alpha \in \mathbb{N}_0^s. \quad (5)$$

With the *Pochhammer symbols* or *falling factorials*

$$(\cdot)_\alpha := \prod_{j=1}^s \prod_{k=0}^{\alpha_j-1} ((\cdot)_j - k), \quad (6)$$

the inverse of  $L$  takes the form

$$L^{-1}p := \sum_{|\gamma| \leq \deg p} \frac{1}{\gamma!} D^\gamma p(0) (\cdot)_\gamma, \quad (7)$$

see again [15]. The *Stirling numbers of first kind*

$$\left[ \begin{smallmatrix} \nu \\ \kappa \end{smallmatrix} \right] := \frac{1}{\kappa!} (D^\kappa (\cdot)_\nu)(0), \quad (8)$$

allow us to express the inverse  $L^{-1}$  in analogous way for the coefficients of the representation  $p = \sum_\alpha \hat{p}_\alpha (\cdot)_\alpha$ . Indeed,

$$(L^{-1}p)_\alpha = \sum_{\beta \in \mathbb{N}_0^s} \left[ \begin{smallmatrix} \beta \\ \alpha \end{smallmatrix} \right] \hat{p}_\beta.$$

By the Newton interpolation formula for integer sites, cf. [10,19], and the Taylor formula we then get

$$\begin{aligned} (\cdot)^\alpha &= \sum_{\beta \leq \alpha} \frac{1}{\beta!} (\Delta^\beta (\cdot)^\alpha) (0) (\cdot)_\beta = \sum_{\beta \in \mathbb{N}_0^s} \left\{ \begin{matrix} \alpha \\ \beta \end{matrix} \right\} (\cdot)_\beta \\ &= \sum_{\beta \in \mathbb{N}_0^s} \left\{ \begin{matrix} \alpha \\ \beta \end{matrix} \right\} \sum_{\gamma \leq \beta} \frac{1}{\gamma!} (D^\gamma (\cdot)_\beta) (0) (\cdot)^\gamma = \sum_{\beta, \gamma \in \mathbb{N}_0^s} \left\{ \begin{matrix} \alpha \\ \beta \end{matrix} \right\} \left[ \begin{matrix} \beta \\ \gamma \end{matrix} \right] (\cdot)^\gamma \end{aligned}$$

from which a comparison of coefficients yields the extension of the well-known duality between the Stirling numbers of the two kinds to the multivariate case:

$$\sum_{\beta \in \mathbb{N}_0^s} \left\{ \begin{matrix} \alpha \\ \beta \end{matrix} \right\} \left[ \begin{matrix} \beta \\ \gamma \end{matrix} \right] = \delta_{\alpha, \gamma}, \quad \alpha, \gamma \in \mathbb{N}_0^s. \quad (9)$$

Moreover, the multivariate Sterling numbers satisfy a recurrence similar to the univariate case. To that end, note that the Leibniz rule for the forward difference, cf. [2], yields

$$\Delta^\kappa (\cdot)^{\nu + \epsilon_j} = \Delta^\kappa ((\cdot)^\nu (\cdot)^{\epsilon_j}) = \kappa_j \Delta^\kappa (\cdot)^\nu + \Delta^{\kappa - \epsilon_j} (\cdot)^\nu,$$

which we substitute into (3) to obtain the recurrence

$$\left\{ \begin{matrix} \nu + \epsilon_j \\ \kappa \end{matrix} \right\} = \frac{1}{\kappa!} (\kappa_j \Delta^\kappa (\cdot)^\nu + \Delta^{\kappa - \epsilon_j} (\cdot)^\nu) (0) = \kappa_j \left\{ \begin{matrix} \nu \\ \kappa \end{matrix} \right\} + \left\{ \begin{matrix} \nu \\ \kappa - \epsilon_j \end{matrix} \right\}. \quad (10)$$

The operator  $L$  also can be used to relate structures between polynomial subspaces.

*Remark 1.* Except [18], which however does not connect to the above, I was not able to find references about multivariate Sterling numbers, so the above simple and elementary proofs are added for the sake of completeness. Nevertheless, Gould's statement from [6] may well be true: "... aber es mag von Interesse sein, daß mindestestens tausend Abhandlungen in der Literatur existieren, die sich mit den Stirlingschen Zahlen beschäftigen. Es ist also sehr schwer, etwas Neues über die Stirlingschen Zahlen zu entdecken."

**Definition 1.** A subspace  $\mathcal{P}$  of  $\Pi$  is called shift invariant if

$$p \in \mathcal{P} \quad \Leftrightarrow \quad p(\cdot + \alpha) \in \mathcal{P}, \quad \alpha \in \mathbb{N}_0^s, \quad (11)$$

and it is called  $D$ -invariant if

$$p \in \mathcal{P} \quad \Leftrightarrow \quad D^\alpha p \in \mathcal{P}, \quad \alpha \in \mathbb{N}_0^s. \quad (12)$$

The principal shift- and  $D$ -invariant spaces for a polynomial  $p \in \Pi$  are defined as

$$\mathcal{S}(p) := \text{span} \{p(\cdot + \alpha) : \alpha \in \mathbb{N}_0^s\}, \quad \mathcal{D}(p) := \text{span} \{D^\alpha p : \alpha \in \mathbb{N}_0^s\}, \quad (13)$$

respectively.

**Proposition 1.** *A subspace  $\mathcal{P}$  of  $\Pi$  is shift invariant if and only if  $L\mathcal{P}$  is  $D$ -invariant.*

*Proof.* The direction “ $\Leftarrow$ ” has been shown in [15, Lemma 3], so assume that  $\mathcal{P}$  is shift invariant and consider, for some  $\alpha \in \mathbb{N}_0^s$ ,

$$\begin{aligned} D^\alpha Lp &= D^\alpha \sum_{|\gamma| \leq \deg p} \frac{1}{\gamma!} \Delta^\gamma p(0) (\cdot)^\gamma \\ &= \sum_{\gamma \geq \alpha} \frac{1}{(\gamma - \alpha)!} \Delta^{\gamma - \alpha} (\Delta^\alpha p)(0) (\cdot)^{\gamma - \alpha} = L\Delta^\alpha p, \end{aligned}$$

where  $\Delta^\alpha p \in \mathcal{P}$  since the space is shift invariant. Hence  $D^\alpha Lp \in L\mathcal{P}$  which proves that this space is indeed  $D$ -invariant.  $\square$

A simple and well-known consequence of Proposition 1 can be recorded as follows.

**Corollary 1.** *A subspace  $\mathcal{P}$  of  $\Pi$  is invariant under integer shifts if and only if it is invariant under arbitrary shifts.*

*Proof.* If together with  $p$  also all  $p(\cdot + \alpha)$  belong to  $\mathcal{P}$  then, by Proposition 1, the space  $L\mathcal{P}$  is  $D$ -invariant from which it follows by [15, Lemma 3] that  $p \in \mathcal{P} = L^{-1}L\mathcal{P}$  implies  $p(\cdot + y) \in \mathcal{P}$ ,  $y \in \mathbb{C}^s$ .

**Proposition 2.** *For  $q \in \Pi$  we have that  $LS(q) = \mathcal{D}(Lq)$ .*

*Proof.* By Proposition 1,  $LS(q)$  is a  $D$ -invariant space that contains  $Lq$ , hence  $LS(q) \supseteq \mathcal{D}(Lq)$ . On the other hand  $L^{-1}\mathcal{D}(Lq)$  is a shift invariant space containing  $Lq$ , hence

$$L^{-1}\mathcal{D}(Lq) \supseteq \mathcal{S}(L^{-1}Lq) = \mathcal{S}(q),$$

and applying the invertible operator  $L$  to both sides of the inclusion yields that  $LS(q) \subseteq \mathcal{D}(Lq)$  and completes the proof.  $\square$

Stirling numbers do not only relate invariant spaces, they also are useful for studying another popular differential operator. To that end, we define the partial differential operators

$$\frac{\hat{\partial}}{\hat{\partial}x_j} = (\cdot)_j \frac{\partial}{\partial x_j} \quad \text{and} \quad \hat{D}^\alpha := \frac{\hat{\partial}^\alpha}{\hat{\partial}x^\alpha}, \quad \alpha \in \mathbb{N}_0^s, \quad (14)$$

also known as  $\theta$ -operator in the univariate case. Recall that the multivariate  $\theta$ -operator is usually of the form

$$\sum_{|\alpha|=n} \hat{D}^\alpha$$

and its eigenfunctions are the homogeneous polynomials, the associated eigenvalues is their total degree. Here, however, we need the partial  $\theta$ -operators. To relate differential operators based on  $\hat{D}$  to standard differential operators, we use the notation  $(\xi D)^\alpha := \xi^\alpha D^\alpha$  for the  $\xi$  scaled partial derivatives,  $\xi \in \mathbb{C}^s$  and use, as common,  $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$ .

**Theorem 1.** *For any  $q \in \Pi$  and  $\xi \in \mathbb{C}^s$  we have that*

$$(q(\hat{D}))p(\xi) = (Lq(\xi D))p(\xi), \quad p \in \Pi. \quad (15)$$

*Proof.* We prove by induction that

$$\hat{D}^\alpha = \sum_{\beta \leq \alpha} \left\{ \begin{matrix} \alpha \\ \beta \end{matrix} \right\} (\cdot)^\beta D^\beta, \quad \alpha \in \mathbb{N}_0^s, \quad (16)$$

which is trivial for  $\alpha = 0$ . The inductive step uses the Leibniz rule to show that

$$\begin{aligned} \hat{D}^{\alpha+\epsilon_j} &= x_j \frac{\partial}{\partial x_j} \sum_{\beta \leq \alpha} \left\{ \begin{matrix} \alpha \\ \beta \end{matrix} \right\} (\cdot)^\beta D^\beta = \sum_{\beta \leq \alpha} \left\{ \begin{matrix} \alpha \\ \beta \end{matrix} \right\} (\beta_j (\cdot)^\beta D^\beta + (\cdot)^{\beta+\epsilon_j} D^{\beta+\epsilon_j}) \\ &= \sum_{\beta \leq \alpha+\epsilon_j} \left( \beta_j \left\{ \begin{matrix} \alpha \\ \beta \end{matrix} \right\} + \left\{ \begin{matrix} \alpha \\ \beta - \epsilon_j \end{matrix} \right\} \right) (\cdot)^\beta D^\beta, \end{aligned}$$

from which (16) follows by taking into account the recurrence (10). Thus, by (5),

$$q(\hat{D}) = \sum_{\alpha \in \mathbb{N}_0^s} q_\alpha \hat{D}^\alpha = \sum_{\alpha \in \mathbb{N}_0^s} q_\alpha \sum_{\beta \in \mathbb{N}_0^s} \left\{ \begin{matrix} \alpha \\ \beta \end{matrix} \right\} (\cdot)^\beta D^\beta = \sum_{\beta \in \mathbb{N}_0^s} (Lq)_\beta (\cdot)^\beta D^\beta,$$

and by applying the differential operator to  $p$  and evaluating at  $\xi$ , we obtain (15).  $\square$

### 3 Ideals and Hermite interpolation

A set  $I \subseteq \Pi$  of polynomials is called an *ideal* in  $\Pi$  if it is closed under addition and multiplication with arbitrary polynomials. A projection  $P : \Pi \rightarrow \Pi$  is called an *ideal projector*, cf. [3], if  $\ker P := \{p \in \Pi : Pp = 0\}$  is an ideal. Ideal projectors with finite range are *Hermite interpolants*, that is, projections  $H : \Pi \rightarrow \Pi$  such that

$$(q(D)Hp)(\xi) = q(D)p(\xi), \quad q \in \mathcal{Q}_\xi, \quad \xi \in \Xi, \quad (17)$$

where  $\mathcal{Q}_\xi$  is a finite dimensional  $D$ -invariant subspace of  $\Pi$  and  $\Xi \subset \mathbb{C}^s$  is a finite set, cf [11]. A polynomial  $p \in \ker H$  vanishes at  $\xi \in \Xi$  with *multiplicity*  $\mathcal{Q}_\xi$ , see [8] for a definition of multiplicity of the common zero of a set of polynomials as a structured quantity.

A particular case is that  $\mathcal{Q}_\xi$  is a *principal  $D$ -invariant space* of the form  $\mathcal{Q}_\xi = \mathcal{D}(q_\xi)$  for some  $q_\xi \in \Pi$ , i.e., the multiplicities are generated by a single polynomial. We say that the respective Hermite interpolation problem and the associated ideal are of *principal multiplicity* in this case. By means of the differential operator  $\hat{D}$  these ideals are also created by shift invariant spaces.

**Theorem 2.** For a finite  $\Xi \subset \mathbb{C}^s$  and polynomials  $q_\xi \in \Pi$ ,  $\xi \in \Xi$ , the polynomial space

$$\left\{ p \in \Pi : q(\hat{D})p(\xi) = 0, q \in \mathcal{S}(q_\xi), \xi \in \Xi \right\} \quad (18)$$

is an ideal of principal multiplicity. Conversely, if  $\xi \in \mathbb{C}_*^s$  then any ideal of principal multiplicity can be written in the form (18).

*Proof.* For  $\xi \in \Xi$  we set  $\mathcal{Q}'_\xi = \mathcal{D}(Lq_\xi)$  which equals  $LS(q_\xi)$  by Proposition 2. Then, also

$$\mathcal{Q}_\xi := \{q(\text{diag } \xi \cdot) : q \in \mathcal{Q}'_\xi\}$$

is a  $D$ -invariant space generated by  $Lq_\xi(\text{diag } \xi \cdot)$ , and by Theorem 1 it follows that

$$q(\hat{D})p(\xi) = 0, \quad q \in \mathcal{S}(q_\xi)$$

if and only if

$$q(D)p(\xi) = 0, \quad q \in \mathcal{Q}_\xi = \mathcal{D}(Lq_\xi(\text{diag } \xi \cdot)).$$

This proves the first claim, the second one follows from the observation that the process is reversible provided that  $\text{diag } \xi$  is invertible which happens if and only if  $\xi \in \mathbb{C}_*^s$ .  $\square$

The *Hermite interpolation problem* based on  $\Xi$  and polynomials  $q_\xi$  can now be phrased as follows: given  $g \in \Pi$  find a polynomial  $p$  (in some prescribed space) such that

$$q(\hat{D})p(\xi) = q(\hat{D})g(\xi), \quad q \in \mathcal{S}(q_\xi), \quad \xi \in \Xi. \quad (19)$$

Clearly, the number of interpolation conditions for this problem is the *total multiplicity*

$$N = \sum_{\xi \in \Xi} \dim \mathcal{S}(q_\xi).$$

The name “multiplicity” is justified here since  $\dim \mathcal{Q}_\xi$  is the scalar multiplicity of a common zero of a set of polynomials and  $N$  counts the total multiplicity. Note however, that this information is incomplete since problems with the same  $N$  can nevertheless be structurally different.

*Example 1.* Consider  $q_\xi(x) = x_1x_2$  and  $q_\xi(x) = x_1^3$ . In both cases  $\dim \mathcal{S}(q_\xi) = 4$  although, of course, the spaces  $\text{span}\{1, x_1, x_2, x_1x_2\}$  and  $\text{span}\{1, x_1, x_1^2, x_1^3\}$  do not coincide.

A subspace  $\mathcal{P} \subset \Pi$  of polynomials is called an *interpolation space* if for any  $g \in \Pi$  there exists  $p \in \mathcal{P}$  such that (19) is satisfied. A subspace  $\mathcal{P}$  is called a *universal interpolation space* of order  $N$  if this is possible for *any* choice of  $\Xi$  and  $q_\xi$  such that

$$\sum_{\xi \in \Xi} \dim \mathcal{S}(q_\xi) \leq N.$$

Using the definition

$$\mathcal{I}_n := \left\{ \alpha \in \mathbb{N}_0^s : \prod_{j=1}^s (1 + \alpha_j) \leq n \right\}, \quad n \in \mathbb{N},$$

of the *first hyperbolic orthant*, the positive part of the *hyperbolic cross*, we can give the following statement that also tells us that the Hermite interpolation problem is always solvable.

**Theorem 3.**  $\Pi_{\mathcal{I}_N}$  is a universal interpolation space for the interpolation problem (19).

*Proof.* Since the interpolant to (19) is an ideal projector by Theorem 2, its kernel, the set of all homogeneous solutions to (19), forms a zero dimensional ideal in  $\Pi$ . This ideal has a Gröbner basis, for example with respect to the graded lexicographical ordering, cf. [5], and the remainders of division by this basis form the space  $\Pi_A$  for some lower set  $A \subset \mathbb{N}_0^s$  of cardinality  $N$ . Since  $\mathcal{I}_N$  is the union of all lower sets of cardinality  $\leq N$ , it contains  $\Pi_A$  and therefore  $\Pi_{\mathcal{I}_N}$  is a universal interpolation space.  $\square$

## 4 Application to the generalized Prony problem

We now use the tools of the preceding sections to investigate the structure of the generalized Prony problem (1) and to show how to reconstruct  $\Omega$  and the polynomials  $f_\omega$  from integer samples. As in [16,17] we start by considering for  $A, B \subset \mathbb{N}_0^s$  the *Hankel matrix*

$$F_{A,B} = \left[ f(\alpha + \beta) : \begin{array}{l} \alpha \in A \\ \beta \in B \end{array} \right] \quad (20)$$

of samples.

*Remark 2.* Instead of the Hankel matrix  $F_{A,B}$  one might also consider the *Toeplitz matrix*

$$T_{A,B} = \left[ f(\alpha - \beta) : \begin{array}{l} \alpha \in A \\ \beta \in B \end{array} \right], \quad A, B \subset \mathbb{N}_0^s, \quad (21)$$

which would lead to essentially the same results. The main difference is the set on which  $f$  is sampled, especially if  $A, B$  are chosen as the total degree index sets  $\Gamma_n := \{\alpha \in \mathbb{N}_0^s : |\alpha| \leq n\}$  for some  $n \in \mathbb{N}$ .

Given a finite set  $\Theta \subset \Pi'$  of linearly independent linear functionals on  $\Pi$  and  $A \subset \mathbb{N}_0^s$  the monomial *Vandermonde matrix* for the interpolation problem at  $\Theta$  is defined as

$$V(\Theta, A) := \left[ \theta(\cdot)^\alpha : \begin{array}{l} \theta \in \Theta \\ \alpha \in A \end{array} \right]. \quad (22)$$



It is standard linear algebra to show that the interpolation problem

$$\Theta p = y, \quad y \in \mathbb{C}^\Theta, \quad \text{i.e.} \quad \theta p = y_\theta, \quad \theta \in \Theta, \quad (23)$$

has a solution for any data  $y \in \mathbb{C}^\Theta$  iff  $\text{rank } V(\Theta, A) \geq \#\Theta$  and that the solution is unique iff  $V(\Theta, A)$  is a nonsingular, hence square, matrix.

For our particular application, we choose  $\Theta$  in the following way: Let  $Q_\omega$  be a basis for the space  $\mathcal{S}(f_\omega)$  and set

$$\Theta_\Omega := \bigcup_{\omega \in \Omega} \{\theta_\omega q(\hat{D}) : q \in Q_\omega\}, \quad \theta_\omega p := p(e^\omega).$$

Since

$$(\Delta^\alpha f_\omega : |\alpha| = \deg f)$$

is a nonzero vector of complex numbers or constant polynomials, we know that  $1 \in \mathcal{S}(f_\omega)$  and will therefore always make the assumption that  $1 \in Q_\omega$ ,  $\omega \in \Omega$ , which corresponds to  $\theta_\omega \in \Theta_\Omega$ ,  $\omega \in \Omega$ . Moreover, we request without loss of generality that  $f_\omega \in Q_\omega$ .

We pattern the Vandermonde matrix conveniently as

$$V(\Theta_\Omega, A) = \left[ \left( q(\hat{D})(\cdot)^\alpha \right) (e^\omega) : \begin{array}{l} q \in Q_\omega, \omega \in \Omega \\ \alpha \in A \end{array} \right]$$

to obtain the following fundamental factorization of the Hankel matrix.

**Theorem 4.** *The Hankel matrix  $F_{A,B}$  can be factored into*

$$F_{A,B} = V(\Theta_\Omega, A)^T F V(\Theta_\Omega, B), \quad (24)$$

where  $F$  is a nonsingular block diagonal matrix.

*Proof.* We begin with an idea by Gröbner [8], see also [15], and first note that any  $g \in Q_\omega$  can be written as

$$g(x+y) = \sum_{q \in Q_\omega} c_q(y) q(x), \quad c_q : \mathbb{C}^s \rightarrow \mathbb{C}.$$

Since  $g(x+y)$  also belongs to  $\text{span } Q_\omega$  as a function in  $y$  for fixed  $x$ , we conclude that  $c_q(y)$  can also be written in terms of  $Q_\omega$  and thus have obtained the *linearization formula*

$$g(x+y) = \sum_{q, q' \in Q_\omega} a_{q, q'}(g) q(x) q'(y), \quad a_{q, q'}(g) \in \mathbb{C}, \quad (25)$$

from [8]. Now consider

$$(F_{A,B})_{\alpha, \beta} = f(\alpha + \beta) = \sum_{\omega \in \Omega} f_\omega(\alpha + \beta) e^{\omega^T(\alpha + \beta)}$$

$$\begin{aligned}
&= \sum_{\omega \in \Omega} \sum_{q, q' \in Q_\omega} a_{q, q'}(f_\omega) q(\alpha) e^{\omega^T \alpha} q'(\beta) e^{\omega^T \beta} \\
&= \sum_{\omega \in \Omega} \sum_{q, q' \in Q_\omega} a_{q, q'}(f_\omega) \left( q(\hat{D})(\cdot)^\alpha \right) (e^\omega) \left( q'(\hat{D})(\cdot)^\beta \right) (e^\omega) \\
&= \left( V(\Theta_\Omega, A) \operatorname{diag} \left( \left[ a_{q, q'}(f_\omega) : \begin{matrix} q \in Q_\omega \\ q' \in Q_\omega \end{matrix} \right] : \omega \in \Omega \right) V(\Theta_\Omega, A)^T \right)_{\alpha, \beta} \\
&=: V(\Theta_\Omega, A)^T F V(\Theta_\Omega, B),
\end{aligned}$$

which already yields (26).

It remains to prove that the blocks  $A_\omega := \left[ a_{q, q'}(f_\omega) : \begin{matrix} q \in Q_\omega \\ q' \in Q_\omega \end{matrix} \right]$  of  $F$  are nonsingular. To that end, we recall that  $f_\omega \in Q_\omega$ , hence, by (25),

$$f_\omega = f_\omega(\cdot + 0) = \sum_{q, q' \in Q_\omega} a_{q, q'}(f_\omega) q(x) q'(0),$$

that is, by linear independence of the elements of  $Q_\omega$ ,

$$\sum_{q' \in Q_\omega} a_{q, q'}(f_\omega) q'(0) = \delta_{q, f_\omega}, \quad q \in Q_\omega,$$

which can be written as  $A_\omega Q_\omega(0) = e_{f_\omega}$  where  $Q_\omega$  also stands for the polynomial vector formed by the basis elements. Since  $Q_\omega$  is a basis for  $\mathcal{S}(f_\omega)$ , there exist finitely supported sequences  $c_q : \mathbb{N}_0^s \rightarrow \mathbb{C}$ ,  $q \in Q_\omega$ , such that

$$q = \sum_{\alpha \in \mathbb{N}_0^s} c_q(\alpha) f(\cdot + \alpha) = \sum_{q', q'' \in Q_\omega} a_{q', q''}(f_\omega) \left( \sum_{\alpha \in \mathbb{N}_0^s} c_q(\alpha) q''(\alpha) \right) q'$$

from which a comparison of coefficients allows us to conclude that

$$A_\omega \sum_{\alpha \in \mathbb{N}_0^s} c_q(\alpha) Q_\omega(\alpha) = e_q, \quad q \in Q_\omega,$$

which even gives an “explicit” formula for the columns of  $A_\omega^{-1}$ .  $\square$

*Remark 3.* If  $F_{A, B}$  is replaced by the Toeplitz matrix from (21), then the factorization becomes

$$T_{A, B} = W(\Theta_\Omega, A) F W(\Theta_\Omega, B)^*, \quad W(\Theta_\Omega, A) := V(\Theta_\Omega, A)^T \quad (26)$$

which has more similarity to a block Schur decomposition since now a Hermitian of the factorizing matrix appears.

Once the factorization (26) is established, the results from [16, 17] can be applied literally and extend to the case of exponential polynomial reconstruction directly. In particular, the following observation is relevant for the termination of the algorithms. It says that if the row index set  $A$  is “sufficiently rich”, then the full information about the ideal  $I_\Omega := \ker \Theta_\Omega$  can be extracted from the Hankel matrix  $F_{A, B}$ .

**Theorem 5.** *If  $\Pi_A$  is an interpolation space for  $\Theta_\Omega$ , for example if  $A = \Upsilon_N$ , then*

1. *the function  $f$  can be reconstructed from samples  $f(A + B)$ ,  $A, B \subset \mathbb{N}_0^s$ , if and only if  $\Pi_A$  and  $\Pi_B$  are interpolation spaces for  $\Theta_\Omega$ .*
2. *a vector  $p \in \mathbb{C}^B \setminus \{0\}$  satisfies*

$$F_{A,B}p = 0 \quad \Leftrightarrow \quad \sum_{\beta \in B} p_\beta (\cdot)^\beta \in I_\Omega \cap \Pi_B.$$

3. *the mapping  $n \mapsto \text{rank } F_{A,\Gamma_n}$  is the affine Hilbert function for the ideal  $I_\Omega$ .*

Theorem 5 suggests the following generic algorithm: use a nested sequence  $B_0 \subset B_1 \subset B_2 \subset \dots$  of index sets in  $\mathbb{N}_0^s$  such that there exist  $j(n) \in \mathbb{N}$ ,  $n \in \mathbb{N}$ , such that  $B_{j(n)} = \Gamma_n$ . In other words: the subsets progress in a *graded* fashion. Then, for  $j = 0, 1, \dots$

1. Consider the kernel of  $F_{\Upsilon_N, B_j}$ , these are the ideal elements in  $\Pi_{B_j}$ .
2. Consider the complement of the kernel, these are elements of the *normal set* and eventually form a basis for an interpolation space.
3. Terminate if  $\text{rank } F_{\Upsilon_N, B_{j(n+1)}} = \text{rank } F_{\Upsilon_N, B_{j(n)}}$  for some  $n$ .

Two concrete instances of this approach were presented and discussed earlier: [16] uses  $B_j = \Gamma_j$  and Sparse Homogeneous Interpolation Techniques (DNSIN) to compute an orthonormal H-basis and a graded basis for the ideal and the normal space, respectively. Since these computations are based on *orthogonal decompositions*, mainly *QR* factorizations, it is numerically stable and suitable for finite precision computations in a floating point environment. A symbolic approach where the  $B_j$  are generated by adding multiindices according to a graded term order, thus using Sparse Monomial Interpolation with Least Elements (SMILE), was introduced in [17]. This method is more efficient in terms of number of computations and therefore suitable for a symbolic framework with exact rational arithmetic.

*Remark 4.* The only a priori knowledge these algorithms need to know is an upper estimate for the *multiplicity*  $N$ .

It should be mentioned that also [13] gives algorithms to reconstruct frequencies and coefficients by first determining the *Prony ideal*  $I_\Omega$ ; the way how these algorithms work and how they are derived are different, however. It would be worthwhile to study and understand the differences between and the advantages of the methods.

While we will point out in the next section how the frequencies can be determined by generalized eigenvalue methods, we still need to clarify how the coefficients of the polynomials  $f_\omega$  can be computed once the ideal structure and the frequencies are determined. To that end, we write

$$f_\omega = \sum_{\alpha \in \mathbb{N}_0^s} f_{\omega, \alpha} (\cdot)^\alpha$$

and note that, with  $\xi_\omega := e^\omega \in \mathbb{C}_*^s$

$$f(\beta) = \sum_{\omega \in \Omega} f_\omega(\beta) e^{\omega^T \beta} = \sum_{\omega \in \Omega} \sum_{\alpha \in \mathbb{N}_0^s} f_{\omega, \alpha} \beta^\alpha \xi_\omega^\beta = \sum_{\omega \in \Omega} \sum_{\alpha \in \mathbb{N}_0^s} f_{\omega, \alpha} \left( \hat{D}^\alpha(\cdot)^\beta \right) (\xi_\omega).$$

In other words, we have for any choice of  $A_\omega \subset \mathbb{N}_0^s$ ,  $\omega \in \Omega$  and  $B \subset \mathbb{N}_0^s$  that

$$\begin{aligned} f(B) &:= [f(\beta) : \beta \in B] \\ &= \left[ \left( \hat{D}^\alpha(\cdot)^\beta \right) (\xi_\omega) : \begin{array}{c} \beta \in B \\ \alpha \in A_\omega, \omega \in \Omega \end{array} \right] [f_{\omega, \alpha} : \alpha \in A_\omega, \omega \in \Omega] \\ &=: G_{A, B} f_\Omega. \end{aligned}$$

The matrix  $G_{A, B}$  is another Vandermonde matrix for a Hermite-type interpolation problem with the functionals

$$\theta_\omega \hat{D}^\alpha, \quad \alpha \in A_\omega, \omega \in \Omega. \quad (27)$$

The linear system

$$G_{A, B} f_\Omega = f(B)$$

can thus be used to determine  $f_\Omega$ : first note that  $L\Pi_n = \Pi_n$  and therefore it follows by Theorem 2 that the interpolation problem is a Hermite problem, i.e., its kernel is an ideal. If we set

$$N = \sum_{\omega \in \Omega} \binom{\deg f_\omega + s}{s} - 1$$

then, by Theorem 3, the space  $\Pi_{\mathcal{R}_N}$  is a universal interpolation space for the interpolation problem (27). Hence, with  $A_\omega = \Gamma_{\deg f_\omega}$ , the matrix  $G_{A, \mathcal{R}_N}$  contains a nonsingular square matrix of size  $\#A \times \#A$  and the coefficient vector  $f_\Omega$  is the unique solution of the overdetermined interpolation problem.

*Remark 5.* The a priori information about the multiplicity  $N$  of the interpolation points does not allow for an efficient reconstruction of the frequencies as it only says that there are at most  $N$  points or points of local multiplicity up to  $N$ .

Nevertheless, the degrees  $\deg f_\omega$ ,  $\omega \in \Omega$ , more precisely, upper bounds for them, can be derived as a by-product of the determination of the frequencies  $\omega$  by means of multiplication tables. To clarify this relationship, we briefly revise the underlying theory, mostly due to Möller and Stetter [12], in the next section.

## 5 Multiplication tables and multiple zeros

Having computed a good basis  $H$  for the ideal  $I_\Omega$  and a basis for the normal set  $\Pi/I_\Omega$ , the final step consists of finding the common zeros of  $H$ . The method of choice is still to use eigenvalues of the multiplication tables, cf. [1, 20], but things become slightly more intricate since we now have to consider the case of zeros with multiplicities, cf. [12].

Let us briefly recall the setup in our particular case. The multiplicity space at  $\xi_\omega = e^\omega \in \mathbb{C}_*^s$  is

$$\mathcal{Q}_\omega := \mathcal{D}(Lf_\omega(\text{diag } \xi_\omega \cdot))$$

and since this is a  $D$ -invariant subspace, it has a graded basis  $\mathcal{Q}_\omega$  where the highest degree element in this basis can be chosen as  $g_\omega := Lf_\omega(\text{diag } \xi_\omega \cdot)$ . Since  $\mathcal{Q}_\omega = \mathcal{D}(g_\omega)$ , all other basis elements  $q \in \mathcal{Q}_\omega$  can be written as  $q = g_q(D)g_\omega$ ,  $g_q \in \Pi$ ,  $q \in \mathcal{Q}_\Omega$ .

Given a basis  $P$  of the normal set  $\Pi/I_\Omega$  and a normal form operator  $\nu : \Pi \rightarrow \Pi/I_\Omega = \text{span } P$  modulo  $I_\Omega$  (which is an ideal projector and can be computed efficiently for Gröbner and H-bases), the multiplication  $p \mapsto \nu((\cdot)_j p)$  is a linear operation on  $\Pi/I_\Omega$  for any  $j = 1, \dots, s$ . It can be represented with respect to the basis  $P$  by means of a matrix  $M_j$  which is called  $j$ th *multiplication table* and gives the multivariate generalization of the *Frobenius companion matrix*.

Due to the unique solvability of the Hermite interpolation problem in  $\Pi/I_\Omega$ , there exists a basis of fundamental polynomials  $\ell_{\omega,q}$ ,  $q \in \mathcal{Q}_\omega$ ,  $\omega \in \Omega$ , such that

$$q'(D)\ell_{\omega,q}(\xi_{\omega'}) = \delta_{\omega,\omega'}\delta_{q,q'}, \quad q' \in \mathcal{Q}_{\omega'}, \quad \omega' \in \Omega. \quad (28)$$

The projection to the normal set, i.e., the interpolant, can now be written for any  $p \in \Pi$  as

$$Lp = \sum_{\omega \in \Omega} \sum_{q \in \mathcal{Q}_\omega} q(D)p(\xi_\omega) \ell_{\omega,q}$$

hence, by the Leibniz rule and the fact that  $\mathcal{Q}_\omega$  is  $D$ -invariant

$$\begin{aligned} L((\cdot)_j \ell_{\omega,q}) &= \sum_{\omega' \in \Omega} \sum_{q' \in \mathcal{Q}_{\omega'}} q'(D)((\cdot)_j \ell_{\omega,q})(\xi_{\omega'}) \ell_{\omega',q'} \\ &= \sum_{\omega' \in \Omega} \sum_{q' \in \mathcal{Q}_{\omega'}} \left( (\cdot)_j q'(D)\ell_{\omega,q}(\xi_{\omega'}) + \frac{\partial q'}{\partial x_j}(D)\ell_{\omega,q}(\xi_{\omega'}) \right) \ell_{\omega',q'} \\ &= (\cdot)_j \ell_{\omega,q} + \sum_{q' \in \mathcal{Q}_\omega} \sum_{q'' \in \mathcal{Q}_\omega} c_j(q', q'') q''(D)\ell_{\omega,q}(\xi_\omega) \ell_{\omega,q'} \\ &= (\cdot)_j \ell_{\omega,q} + \sum_{q' \in \mathcal{Q}_\omega} c_j(q, q') \ell_{\omega,q'}, \end{aligned} \quad (29)$$

where the coefficients  $c_j(q, q')$  are defined by the expansion

$$\frac{\partial q}{\partial x_j} = \sum_{q' \in \mathcal{Q}_\omega} c_j(q', q) q', \quad q \in \mathcal{Q}_\Omega. \quad (30)$$

Note that the coefficients in (30) are zero if  $\deg q' \geq \deg q$ . Therefore  $c_j(q, q') = 0$  in (29) if  $\deg q' \geq \deg q$ . In particular, since  $g_\omega$  is the unique element of maximal degree in  $\mathcal{Q}_\omega$ , it we have that

$$L((\cdot)_j \ell_{\omega,g_\omega}) = (\cdot)_j \ell_{\omega,g_\omega}, \quad \omega \in \Omega. \quad (31)$$

This way, we have given a short and simple proof of the following result from [12], restricted to our special case of principal multiplicities.

**Theorem 6.** *The eigenvalues of the multiplication tables  $M_j$  are the components of the zeros  $(\xi_\omega)_j$ ,  $\omega \in \Omega$ , the associated eigenvectors the polynomials  $\ell_{\omega, g_\omega}$  and the other fundamental polynomials form an invariant space.*

In view of numerical linear algebra, the eigenvalue problems for ideals with multiplicities become unpleasant as in general the matrices become derogatory, except when  $g_\omega$  is a power of linear function, i.e.,  $g_\omega = (v^T \cdot)^{\deg g_\omega}$  for some  $v \in \mathbb{R}^s$ .

There is, however, a remedy described in [4, p. 48]: building a matrix from traces of certain multiplication tables, one can construct a basis for the associated radical ideal, thus avoiding the hassle with multiplicities. Though this approach is surprisingly elementary, we will not go into details here as it is not in the scope of the paper, but refer once more to the recommendable collection [4].

Moreover, the dimension of the respective invariant spaces is an upper bound for  $\deg f_\omega$  which can help to set up the parameters in the interpolation problem in Section 4.

## 6 Conclusion

The generalized version of Prony's problem with polynomial coefficients is a straightforward extension of the standard problem with constant coefficients. The main difference is that in (1) *multiplicities* of common zeros in an ideal play a role where the multiplicity spaces are related to the shift invariant space generated by the coefficients via the operator  $L$  from (4). This operator which relates the Taylor expansion and interpolation at integer points in the Newton form, has in turn a natural relationship with multivariate Stirling numbers of the second kind. These properties can be used to extend the algorithms from [16,17] almost without changes to the generalized case, at least as far the construction of a good basis for the Prony ideal is concerned.

Implementations, numerical tests and comparison with the algorithms from [13] are straightforward lines of further work and may be a worthwhile waste of time.

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